

On spectral radius of strongly connected digraphs

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Abstract

We determine the digraphs which achieve the second, the third and the fourth minimum spectral radii respectively among strongly connected digraphs of order $n \geq 4$, and thus we answer affirmatively the problem whether the unique digraph which achieves the minimum spectral radius among all strongly connected bicyclic digraphs of order n achieves the second minimum spectral radius among all strongly connected digraphs of order n for $n \geq 4$ proposed in [H. Lin, J. Shu, A note on the spectral characterization of strongly connected bicyclic digraphs, Linear Algebra Appl. 436 (2012) 2524–2530]. We also discuss the strongly connected bicyclic digraphs with small and large spectral radii respectively.

Keywords: spectral radius, strongly connected digraph, bicyclic digraph, non-negative irreducible matrix

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1 Introduction

We consider digraphs without loops and multiple arcs. Let D be a digraph of order n with vertex set $V(D)$ and arc set $E(D)$. Let $V(D) = \{v_1, v_2, \dots, v_n\}$. The adjacency matrix of D is the $(0, 1)$ -matrix $A(D) = (a_{ij})$ of order n where $a_{ij} = 1$ if there is an arc from v_i to v_j , and $a_{ij} = 0$ otherwise. The eigenvalues of D are the eigenvalues of $A(D)$. The spectral radius of D is the largest modulus of an eigenvalue of D , denoted by

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$\rho(D)$. Obviously, the eigenvalues of D are the roots of the characteristic polynomial of D , denoted by $P(D, x)$, defined to be the characteristic polynomial of the matrix $A(D)$, which is $\det(xI_n - A(D))$, where I_n is the identity matrix of order n . The spectra of digraphs have been studied to some extent, see e.g., [1, 3, 4, 5].

A digraph D is strongly connected if for every pair $x, y \in V(D)$, there exists a directed path from x to y and a directed path from y to x . D is called a strongly connected bicyclic digraph if D is strongly connected with $|E(D)| = |V(D)| + 1$. For $n \geq 3$, let \mathbb{B}_n be the set of strongly connected bicyclic digraphs of order n .

Note that D is strongly connected if and only if $A(D)$ is irreducible. It follows from the Perron-Frobenius Theorem that if D is strongly connected, then $\rho(D)$ is an eigenvalue of D and there is a corresponding eigenvector whose coordinates are all positive.

Let P_n be a directed path of order n . If $P_n = u_1 u_2 \dots u_n$, then u_1 is the initial vertex, and u_n is the terminal vertex of P_n . The θ -digraph with parameters a, b and c with $a \leq b$, denoted by $\theta(a, b, c)$, consists of three directed paths P_{a+2} , P_{b+2} and P_{c+2} such that the initial vertex of P_{a+2} and P_{b+2} is the terminal vertex of P_{c+2} , and the initial vertex of P_{c+2} is the terminal vertex of P_{a+2} and P_{b+2} . These three directed paths are called the basic directed paths of $\theta(a, b, c)$. A ∞ -digraph with parameters k and l with $k \leq l$, denoted by $\infty(k, l)$, consists of two directed cycles with exactly a vertex in common.

Recently, Lin and Shu [4] showed that $\theta(0, 1, n-3)$ ($\infty(2, n-1)$, respectively) is the unique digraph in \mathbb{B}_n which achieve the minimum (maximum, respectively) spectral radius for $n \geq 4$. Let C_n be the directed cycle of order n . Note that C_n uniquely achieves the minimum spectral radius among all strongly connected digraphs of order $n \geq 3$. Lin and Shu [4] proposed the following problem.

Problem 1.1 *Is $\theta(0, 1, n-3)$ achieving the second minimum spectral radius among all n -vertex strongly connected digraphs for $n \geq 4$?*

Now we determine the unique digraphs which achieve the second, the third and the fourth minimum spectral radii respectively among strongly connected digraphs of order $n \geq 4$, and thus we answer Problem 1.1 affirmatively. To obtain this, we also determine the unique digraphs in \mathbb{B}_n with the second and the third minimum spectral radii respectively for $n \geq 4$. Finally, we determine the unique digraph in \mathbb{B}_n with the second maximum spectral radius for $n \geq 4$.

2 Preliminaries

We list some lemmas that will be used in our proof.

Lemma 2.1 [4] *For $n \geq 4$, $\theta(0, 1, n-3)$ is the unique digraph in \mathbb{B}_n which achieves the minimum spectral radius, and $\infty(\lfloor \frac{n+1}{2} \rfloor, \lceil \frac{n+1}{2} \rceil)$ is the unique ∞ -digraph in \mathbb{B}_n which achieves the minimum spectral radius.*

The following lemma was proved in [4] for $c \geq 1$. By proof there, it is also true for $c = 0$.

Lemma 2.2 [4] *If $b \geq 1$, then $\rho(\theta(a, b, c)) > \rho(\theta(a, b - 1, c + 1))$. If $a \geq 1$, then $\rho(\theta(a, b, c)) > \rho(\theta(a - 1, b, c + 1))$.*

The Following lemma was given in plain text in [4] and it is a consequence of the well known coefficients theorem for digraphs, see e.g., [2, Theorem 1.2, p. 36].

Lemma 2.3 $P(\theta(a, b, c), x) = x^n - x^a - x^b$ with $n = a + b + c + 2$, and $P(\infty(k, l), x) = x^n - x^{k-1} - x^{l-1}$ with $n = k + l - 1$.

Lemma 2.4 *For $n \geq 4$, $\rho(\infty(\lfloor \frac{n+1}{2} \rfloor, \lceil \frac{n+1}{2} \rceil)) > \rho(\theta(0, 2, n - 4))$.*

Proof. Let $D_1 = \theta(0, 2, n - 4)$ and $D_2 = \infty(\lfloor \frac{n+1}{2} \rfloor, \lceil \frac{n+1}{2} \rceil)$. By Lemma 2.3, $P(D_1, x) = x^n - x^2 - 1$ and $P(D_2, x) = x^n - x^{\lfloor \frac{n-1}{2} \rfloor} - x^{\lceil \frac{n-1}{2} \rceil}$. For $x \geq \rho(D_2) > 1$, $P(D_1, x) - P(D_2, x) = -x^2 - 1 + 2x^{\frac{n-1}{2}} \geq -x^2 - 1 + 2x^2 = x^2 - 1 > 0$ if n is odd and $P(D_1, x) - P(D_2, x) = -x^2 - 1 + x^{\frac{n}{2}-1} + x^{\frac{n}{2}} \geq -x^2 - 1 + x + x^2 = x - 1 > 0$ if n is even. Thus $\rho(D_2) > \rho(D_1)$. \square

Lemma 2.5 $\rho(\theta(0, 2, n - 4))$ *is strictly decreasing for $n \geq 4$.*

Proof. Suppose that $n_1 > n_2 \geq 4$. By Lemma 2.3, $P(\theta(0, 2, n_1 - 4), x) - P(\theta(0, 2, n_2 - 4), x) = x^{n_1} - x^{n_2} > 0$ for $x \geq \rho(\theta(0, 2, n_2 - 4)) > 1$. Thus $\rho(\theta(0, 2, n_1 - 4)) < \rho(\theta(0, 2, n_2 - 4))$. \square

Recall that the spectral radius of a nonnegative irreducible matrix B is larger than that of a principal submatrix of B and it increases when an entry of B increases [6, p. 19, 38]. Thus we have the following well known lemma.

Lemma 2.6 *Let D be a strongly connected digraph and H a strongly connected proper subdigraph of D . Then $\rho(D) > \rho(H)$.*

Lemma 2.7 $\rho(\theta(0, n - 2, 0))$ *is strictly decreasing for $n \geq 4$.*

Proof. Suppose that $n_1 > n_2 \geq 4$. By Lemma 2.3, $P(\theta(0, n_1 - 2, 0), x) - P(\theta(0, n_2 - 2, 0), x) = (x^{n_2} - x^{n_2-2})(x^{n_1-n_2} - 1) > 0$ for $x \geq \rho(\theta(0, n_2 - 2, 0)) > 1$. Thus $\rho(\theta(0, n_1 - 2, 0)) < \rho(\theta(0, n_2 - 2, 0))$. \square

Lemma 2.8 [4] *For $n \geq 4$, $\infty(2, n - 1)$ is the unique digraph in \mathbb{B}_n which achieves the maximum spectral radius, and $\theta(0, n - 2, 0)$ is the unique θ -digraph in \mathbb{B}_n which achieves the maximum spectral radius.*

Lemma 2.9 [4] *If $k \geq 1$, then $\rho(\infty(k - 1, l + 1)) > \rho(\infty(k, l))$.*

3 Results

To determine the unique digraphs with the second, the third and the fourth minimum spectral radii respectively among strongly connected digraphs of order $n \geq 4$, we need first to determine the unique digraphs in \mathbb{B}_n with the second and the third minimum spectral radii respectively for $n \geq 4$.

Theorem 3.1 *For $n \geq 4$, $\theta(1, 1, n - 4)$ and $\theta(0, 2, n - 4)$ are the unique digraphs in \mathbb{B}_n which achieve the second and the third minimum spectral radii respectively.*

Proof. Let $D \in \mathbb{B}_n$ with $D \neq \theta(0, 1, n - 3)$. Then D is a θ -digraph or a ∞ -digraph. Suppose that D is a θ -digraph and $D \neq \theta(1, 1, n - 4)$. By Lemma 2.2, we have $\rho(D) \geq \rho(\theta(0, 2, n - 4))$ with equality only if $D = \theta(0, 2, n - 4)$. By Lemma 2.3, $P(\theta(1, 1, n - 4), x) - P(\theta(0, 2, n - 4), x) = -2x + x^2 + 1 = (x - 1)^2 > 0$ for $x \geq \rho(\theta(0, 2, n - 4)) > 1$. Thus $\rho(D) \geq \rho(\theta(0, 2, n - 4)) > \rho(\theta(1, 1, n - 4))$. If D is a ∞ -digraph, then by Lemmas 2.1 and 2.4,

$$\rho(D) \geq \rho\left(\infty\left(\left\lfloor \frac{n+1}{2} \right\rfloor, \left\lceil \frac{n+1}{2} \right\rceil\right)\right) > \rho(\theta(0, 2, n - 4)).$$

Now the result follows from the first part of Lemma 2.1. \square

Theorem 3.2 *Let D be a strongly connected digraph of order $n \geq 4$ that is neither a bicyclic digraph nor C_n . Then $\rho(D) > \rho(\theta(0, 2, n - 4))$.*

Proof. Let C be a shortest directed cycle in G . Obviously, $V(C) \neq V(D)$. There is a vertex $u \in V(D) \setminus V(C)$ such that there is an arc from u to some vertex, say v , on C . Also, there is a directed path from some vertex on C to u . Let w be a vertex on C such that the distance from w to u in D is as small as possible. Let P be such a directed path. Then C and P have exactly one common vertex w . If $w \neq v$, then D has a proper θ -subdigraph, and if $w = v$, then D has a proper ∞ -subdigraph.

If D has a proper ∞ -subdigraph, say $\infty(k, l)$ with $k + l = n_1 + 1$ and $n_1 \leq n$, then by Lemma 2.6, the second part of Lemma 2.1, and Lemmas 2.4 and 2.5, we have

$$\begin{aligned} \rho(D) &> \rho(\infty(k, l)) \\ &\geq \rho\left(\infty\left(\left\lfloor \frac{n_1+1}{2} \right\rfloor, \left\lceil \frac{n_1+1}{2} \right\rceil\right)\right) \\ &> \rho(\theta(0, 2, n_1 - 4)) \\ &\geq \rho(\theta(0, 2, n - 4)). \end{aligned}$$

Suppose that D has a proper θ -subdigraph, say $\theta(a, b, c)$ with $a + b + c = n_2 - 2$ and $n_2 \leq n$.

Case 1. $n_2 \leq n - 1$. By Lemma 2.6 and the first part of Lemma 2.1, we have

$$\rho(D) > \rho(\theta(a, b, c)) \geq \rho(\theta(0, 1, n_2 - 3)).$$

By Lemma 2.3, $P(\theta(0, 2, n-4), x) - P(\theta(0, 1, n_2-3), x) = x^n - x^{n_2} - x^2 + x = x^{n_2}(x^{n-n_2} - 1) - x(x-1) \geq x^{n_2}(x-1) - x(x-1) = (x^{n_2} - x)(x-1) > 0$ for $x \geq \rho(\theta(0, 1, n_2-3)) > 1$. Thus $\rho(\theta(0, 1, n_2-3)) > \rho(\theta(0, 2, n-4))$. Hence $\rho(D) > \rho(\theta(0, 2, n-4))$.

Case 2. $n_2 = n$ and $\theta(a, b, c) \neq \theta(0, 1, n-3)$ and $\theta(1, 1, n-4)$. By Lemma 2.6, the first part of Lemma 2.1, and Theorem 3.1,

$$\rho(D) > \rho(\theta(a, b, c)) \geq \rho(\theta(0, 2, n-4)).$$

Case 3. $n_2 = n$ and the θ -subdigraph of D can only be $\theta(0, 1, n-3)$ or $\theta(1, 1, n-4)$. Suppose without loss of generality that D has a θ -subdigraph $\theta(0, 1, n-3)$ (the proof is similar if D has a θ -subdigraph $\theta(1, 1, n-4)$). Let vw , vu_1w and $wu'_1u'_2 \dots u'_{n-3}v$ be the basic directed paths of the subdigraph $\theta(0, 1, n-3)$. We consider the possible arc(s) in D (except the arcs in $\theta(0, 1, n-3)$) as follows.

- (i) $wv \notin E(D)$; Otherwise, D has a θ -subdigraph $\theta(0, n-3, 0)$, a contradiction.
- (ii) $u_1v \notin E(D)$ and $wu_1 \notin E(D)$; Otherwise, D has a θ -subdigraph $\theta(0, n-2, 0)$, a contradiction.
- (iii) $u_1u'_k \notin E(D)$ and $u'_{n-k-2}u_1 \notin E(D)$ for $2 \leq k \leq n-3$; Otherwise, D has a θ -subdigraph $\theta(0, k, n-k-2)$, a contradiction.
- (iv) $vu'_k \notin E(D)$ and $u'_{n-k-2}w \notin E(D)$ for $1 \leq k \leq n-3$; Otherwise, D has a θ -subdigraph $\theta(0, k+1, n-k-3)$, a contradiction.
- (v) $u'_kv \notin E(D)$ and $wu'_{n-k-2} \notin E(D)$ for $1 \leq k \leq n-4$; Otherwise, D has a θ -subdigraph $\theta(0, 1, k)$, a contradiction.
- (vi) $u'_lu'_k \notin E(D)$ for $1 \leq k < l \leq n-3$; Otherwise, D has a θ -subdigraph $\theta(0, n-l+k-2, l-k-1)$, a contradiction.
- (vii) $u'_ku'_l \notin E(D)$ for $1 \leq k < l-1 \leq n-4$; Otherwise, D has a θ -subdigraph $\theta(0, 1, n-2-(l-k))$, a contradiction.
- (viii) $\{u_1u'_1, u'_{n-3}u_1\} \not\subseteq E(D)$; Otherwise, D has a θ -subdigraph $\theta(0, 1, n-4)$, a contradiction.

From (i)–(viii), we find that besides these arcs in $\theta(0, 1, n-3)$, D contains one additional arc $u_1u'_1$ or $u'_{n-3}u_1$. Thus D is isomorphic to the digraph D' obtained from $\theta(0, 1, n-3)$ by adding the arc $u_1u'_1$. Besides the empty union and C_n , $\mathcal{C}(D')$ contains two directed cycles on $n-1$ vertices. Thus $P(D, x) = P(D', x) = x^n - 2x - 1$. Obviously, $P(D, 1) < 0$, $P(D, 2) > 0$, and $P(D, x)$ is strictly increasing for $x \geq 1$. Thus $1 < \rho(D) < 2$. Similarly, $1 < \rho(\theta(0, 2, n-4)) < 2$ by Lemma 2.3. Note that $P(\theta(0, 2, n-4), x) - P(D, x) = -x^2 + 2x > 0$ for $1 < x < 2$. Thus $\rho(D) > \rho(\theta(0, 2, n-4))$. \square

From Lemma 2.1 and Theorems 3.1 and 3.2, we have the following theorem.

Theorem 3.3 *Among the strongly connected digraphs of order $n \geq 4$, $\theta(0, 1, n - 3)$, $\theta(1, 1, n - 4)$ and $\theta(0, 2, n - 4)$ are the unique digraphs that achieve the second, the third and the fourth minimum spectral radii respectively.*

Thus we answer Problem 1.1 affirmatively.

Finally, we determine the unique digraphs in \mathbb{B}_n with the second maximum spectral radius for $n \geq 4$.

Theorem 3.4 *For $n \geq 5$, $\infty(3, n - 2)$ for $5 \leq n \leq 7$, and $\theta(0, n - 2, 0)$ for $n = 4$ and $n \geq 8$ are the unique digraphs in \mathbb{B}_n which achieve the second maximum spectral radius.*

Proof. Obviously, $\mathbb{B}_4 = \{\infty(2, 3), \theta(0, 2, 0), \theta(1, 1, 0), \theta(0, 1, 1)\}$. By Lemmas 2.1 and 2.8, and Theorem 3.1, we have $\rho(\infty(2, 3)) > \rho(\theta(0, 2, 0)) > \rho(\theta(1, 1, 0)) > \rho(\theta(0, 1, 1))$. Thus the result for $n = 4$ follows.

Suppose that $n \geq 5$. Let $D \in \mathbb{B}_n$ and $D \neq \infty(3, n - 2), \theta(0, n - 2, 0)$. If D is a θ -digraph, then by the second part of Lemma 2.8, $\rho(D) < \rho(\theta(0, n - 2, 0))$. If D is a ∞ -digraph and $D \neq \infty(2, n - 1)$, then by Lemma 2.9, $\rho(D) < \rho(\infty(3, n - 2))$. Now by the first part of Lemma 2.8, the second maximum spectral radius of digraphs in \mathbb{B}_n is $\max\{\rho(\theta(0, n - 2, 0)), \rho(\infty(3, n - 2))\}$, which is only achieved by $\theta(0, n - 2, 0)$ or $\infty(3, n - 2)$.

If $5 \leq n \leq 7$, then by direct calculation using maple, we have $\rho(\theta(0, n - 2, 0)) < \rho(\infty(3, n - 2))$.

Suppose that $n \geq 8$. Let $\rho = \rho(\theta(0, n - 2, 0))$. Obviously, $\rho > 1$. By Lemma 2.7 and direct calculation using maple, we have $\rho \leq \rho(\theta(0, 6, 0)) = 1.1748 \cdots < 1.175$. For $1 < x < 1.175$, let $h(x) = 1 + x + x^2 - x^3 - x^4$. Since $h'(x) = 1 + 2x - 3x^2 - 4x^3 < 0$, $h(x)$ is strictly decreasing. Thus $h(\rho) > h(1.175) = 0.027265 > 0$. By Lemma 2.3, $\rho^{n-2} = \frac{1}{\rho^2-1}$, and thus

$$\begin{aligned} P(\infty(3, n - 2), \rho) &= P(\infty(3, n - 2), \rho) - P(\theta(0, n - 2, 0), \rho) \\ &= \rho^{n-2} + 1 - \rho^{n-3} - \rho^2 \\ &= (\rho - 1)(\rho^{n-3} - \rho - 1) \\ &= (\rho - 1) \left(\frac{1}{\rho(\rho^2 - 1)} - \rho - 1 \right) \\ &= \frac{h(\rho)}{\rho(\rho + 1)} \\ &> 0. \end{aligned}$$

Obviously, $P(\infty(3, n - 2), 1) = -1 < 0$ and $P(\infty(3, n - 2), x)$ is strictly increasing for $x > 1$. Thus $\rho(\infty(3, n - 2)) < \rho = \rho(\theta(0, n - 2, 0))$. \square

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